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HEAT TRANSFER IN AN ANNULUS WITH VARIABLE CIRCUMFERENTIAL HEAT FLUX†

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Abstract—An analysis of heat transfer in a concentric circular tube annulus with an arbitrarily prescribed heat flux around the periphery of either wall, or both walls, is presented. Solutions have been obtained for the hydrodynamically and thermally fully developed condition for constant heat rate per unit of tube length, for both the laminar and turbulent flow regimes. With these results, the ensuing temperature variation around either wall may be predicted. Contrary to what might be expected, the wall temperature variation is very substantial in turbulent as well as laminar flow. An example shows the importance of this effect.

NOMENCLATURE

a, b, A, B,	Fourier series coefficients;
C_n,	wall conduction parameter;
D_H,	hydraulic diameter;
E,	$\left(1 + \frac{\epsilon_H}{\alpha}\right)$;
f,	friction factor;
g,	circumferential temperature function;
k,	thermal conductivity;
Nu,	Nusselt number;
Pr,	Prandtl number;
q'',	heat flux, positive into the fluid;
r,	radial co-ordinate;
$\bar{r}$,	r/r_o;
r^*,	r_i/r_o;
r_o^+,	$r_o u^*/\nu$;
R_n,	eigenfunction;
Re,	Reynolds number;
$\bar{s}$,	radius of zero shear;
t,	temperature;
Δt_0,	radial temperature function;
u,	velocity;
u^*,	$\sqrt{(g_c \tau_o/\rho)}$;
u^+,	$u/(u_m \sqrt{(f/2)})$;
x,	axial co-ordinate;
y,	distance from wall;
y^+,	$y u^*/\nu$;
α,	thermal diffusivity;
ϵ_H,	eddy diffusivity for heat;
ϵ_M,	eddy diffusivity for momentum;
η_o,	$\frac{\bar{r} - \bar{s}}{1 - \bar{s}}$;
η_i,	$\frac{\bar{s} - \bar{r}}{\bar{s} - r^*}$;
δ,	wall thickness;
θ,	angular co-ordinate;
θ^*,	influence coefficient;
ν,	kinematic viscosity;
τ^*,	ratio of outer to inner wall shear.

Subscripts

ave,	average;
in,	inlet;
a,	annulus;
i,	inner wall;
m,	mean;
n,	harmonic index;
o,	outer wall;
w,	wall.

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INTRODUCTION

THE CIRCULAR tube annulus is one of the more important flow passage geometries for heat-transfer systems, ranking closely behind the circular tube in engineering applications. It has been shown by Reynolds *et al.* [1] that the thermal boundary condition for flow through an annulus can be reduced to four fundamental solutions.

Of the large number of combinations of boundary conditions and flow regimes, the problem to be considered here is only that of fully developed laminar and turbulent flow with the heat flux specified at the wall and constant with tube length. For heat flux uniform around the tube periphery, this case has been treated analytically by Kays and Leung [2] for the concentric annulus for turbulent flow over an extensive range of *Re*, *Pr*, and r^*, and has been substantiated by their experiments with air. The laminar flow counterpart of this problem is covered by Lundberg *et al.* [3] as a special case of the complete laminar flow concentric annulus problem.

The additional condition to be considered here, however, concerns the variation of the heat flux around the periphery of the flow passage. This is of particular importance in a nuclear reactor, where the power distribution across a fuel element produces a variation of heat flux around the periphery of the flow passage. The importance of the variable circumferential heat flux problem has recently been shown by Reynolds [4] for the circular tube. This present paper extends the analysis of Reynolds [4] to include the annular geometry, building upon the annulus solution of Kays and Leung [2] for turbulent flow, and Lundberg *et al.* [3] for laminar flow.

The technique used for the solution is to expand the known peripheral heat flux distribution in a Fourier series. Because the resulting temperature distribution can also be expressed as a Fourier series, the energy equation becomes a set of ordinary differential equations in terms of the eigenfunctions. These eigenfunctions are calculated herein, and may be used with the Fourier coefficients of any peripheral heat flux distribution to obtain the resulting wall temperature distribution.

FORMULATION

To investigate the case of variable heat flux around the periphery of a flow passage, we will restrict our attention to fully developed velocity and temperature fields with constant fluid properties. The system under consideration, along with some of the pertinent nomenclature, is illustrated in Fig. 1. Assuming equal eddy

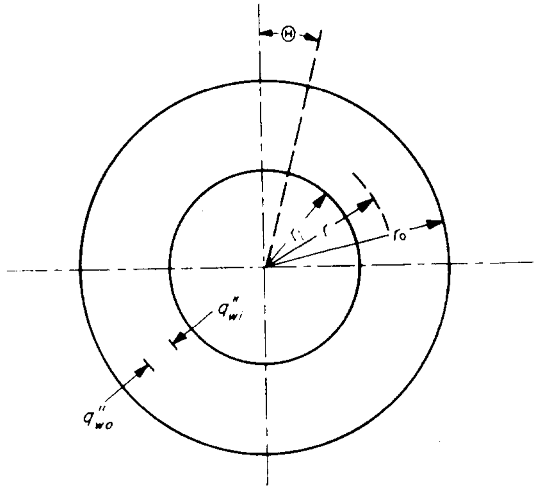

Fig. 1.

diffusivity for heat in the radial and circumferential directions, the governing differential equation becomes

$$L(t) = \frac{1}{r}\frac{\partial}{\partial r}\left[r\,(\alpha + \epsilon_H)\frac{\partial t}{\partial r}\right] + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left[(\alpha + \epsilon_H)\frac{\partial t}{\partial \theta}\right] = u\frac{\partial t}{\partial x}. \quad (1)$$

Following the approach used by Reynolds [4] for the circular tube case, we let

$$t(r, \theta, x) = t_m(x) + \Delta t_o(r) + g(r, \theta).$$

The boundary condition may also be expressed as a sum,

$$q''_w = q'' + F(\theta), \text{ where } \oint F(\theta)\,d\theta = 0.$$

For uniform heat input in the flow direction, and a fully developed temperature profile, the right-hand side of equation (1) is a function of the radius alone,

$$u\frac{dt_m}{dx} = f(r).$$

Equation (1) divides naturally into two parts:

$$L(\Delta t_o) = u \frac{dt_m}{dx}, \qquad (1a)$$

$$L(g) = 0. \qquad (1b)$$

The first equation represents the uniform heat flux case reported by Kays and Leung [2] and by Lundberg *et al.* [3] for the annulus. We need only a solution to the second equation to complete the general case by superposition of solutions.

The specified variation of circumferential heat flux at each wall of the annulus can be expanded in a Fourier series as

$$q''_{wo} = \sum_0^\infty (a_{n_o} \sin n\theta + b_{n_o} \cos n\theta),$$

$$q''_{wi} = \sum_0^\infty (a_{n_i} \sin n\theta + b_{n_i} \cos n\theta).$$

Heat transfer *into* the fluid will be treated as positive, and heat transfer *out* of the fluid will be treated as negative.

If we now let $g(r, \theta)$ be represented by a series expansion,

$$g = g_o + g_i = \frac{D_H}{k} \sum_0^\infty R_{n_o} (a_{n_o} \sin n\theta + b_{n_o} \cos n\theta) + \frac{D_H}{k} \sum_0^\infty R_{n_i} (a_{n_i} \sin n\theta + b_{n_i} \cos n\theta),$$

where g_i is the fluid temperature effect due to heat flux variation around the inner wall with the outer wall insulated, and g_o is the corresponding result for the outer wall variation.

Equation (1b) is then reduced to the dimensionless form,

$$\frac{d}{d\bar{r}}\left\{\bar{r} E \frac{d R_n}{d\bar{r}}\right\} - \frac{n^2 E R_n}{\bar{r}} = 0, \qquad (2)$$

and the boundary conditions become

Heating from inner tube

$$\bar{r} = r^*, \quad \frac{dR_n}{d\bar{r}} = -\frac{r_o}{D_H}$$

$$\bar{r} = 1, \quad \frac{dR_n}{d\bar{r}} = 0$$

Heating from outer tube

$$\bar{r} = r^*, \quad \frac{dR_n}{d\bar{r}} = 0$$

$$\bar{r} = 1, \quad \frac{dR_n}{d\bar{r}} = \frac{r_o}{D_H}$$

The general solution of equation (1a) at the inner and outer walls is [2]

$$\bar{r} = r^*; \quad t_o = t_{wi} - t_m = \frac{D_H q''_i}{k\, Nu_{ii}} \left(1 - \theta_i^* \frac{q''_o}{q''_i}\right) = \frac{D_H q''_i}{k\, Nu_i} = \frac{D_H q''_i}{k} R_{o_i}(r^*) + \frac{D_H q''_o}{k} R_{o_o}(r^*) \qquad (2a)$$

$$\bar{r} = 1; \quad t_o = t_{wo} - t_m = \frac{D_H q''_o}{k\, Nu_{oo}} \left(1 - \theta_o^* \frac{q''_i}{q''_o}\right) = \frac{D_H q''_o}{k\, Nu_o} = \frac{D_H q''_o}{k} R_{o_o}(1) + \frac{D_H q''_i}{k} R_{o_i}(1). \qquad (2b)$$

Combining the solutions of equations (1a) and (1b), the wall temperatures are then

$$t_{wo} - t_m = \frac{D_H}{k} \sum_0^\infty R_{n_o}(1) (a_{n_o} \sin n\theta + b_{n_o} \cos n\theta) + \frac{D_H}{k} \sum_0^\infty R_{n_i}(1) (a_{n_i} \sin n\theta + b_{ni} \cos n\theta), \qquad (3a)$$

and

$$t_{wi} - t_m = \frac{D_H}{k} \sum_0^\infty R_{n_o}(r^*) (a_{n_o} \sin n\theta + b_{n_o} \cos n\theta) + \frac{D_H}{k} \sum_0^\infty R_{n_i}(r^*) (a_{n_i} \sin n\theta + b_{n_i} \cos n\theta). \qquad (3b)$$

The eigenfunctions, R_n, evaluated at the inner or outer wall are reciprocals of Nusselt number at that wall. The index n indicates the harmonic of the Fourier expansion, and the subscript i or o indicates the heated wall.

Wall conduction

Any circumferential wall temperature variation will tend to be attenuated by heat conduction in the wall [5]. Consider a cylindrical wall of finite thickness with an independent heat flux q_s'' imposed on one surface. The heat flux q_w'' resulting on the opposite surface is determined from

$$\frac{\delta}{r^2} k_w \frac{d^2 t}{d\theta^2} - q_w''(\theta) + q_s''(\theta) = 0 \qquad (4)$$

Expanding the imposed heat flux in Fourier series,

$$q_{si}'' = \sum_0^\infty (A_{n_i} \sin n\theta + B_{n_i} \cos n\theta),$$

or

$$q_{so}'' = \sum_0^\infty (A_{n_o} \sin n\theta + B_{n_o} \cos n\theta).$$

Substituting equation (3a) or (3b) into equation (4), we get a relation between $q_w''(\theta)$ and $q_s''(\theta)$. We define the dimensionless groupings

$$\frac{\delta_o k_{wo} D_H}{k r_o^2} R_{n_i}(1) n^2 = C_{n_{oi}},$$

outer wall conduction
from inner wall heating

$$\frac{\delta_o k_{wo} D_H}{k r_o^2} R_{n_o}(1) n^2 + 1 = C_{n_{oo}},$$

outer wall conduction
from outer wall heating

$$\frac{\delta_i k_{wi} D_H}{k r_i^2} R_{ni}(r^*) n^2 + 1 = C_{n_{ii}},$$

inner wall conduction
from inner wall heating

and

$$\frac{\delta_i k_{wi} D_H}{k r_i^2} R_{n_o}(r^*) n^2 = C_{n_{io}}.$$

inner wall conduction
from outer wall heating

Then the heat flux which is actually imposed on the fluid has the following Fourier series coefficients:

$$a_{n_o} = \frac{A_{n_o} C_{n_{ii}} - A_{n_i} C_{n_{oi}}}{C_{n_{oo}} C_{n_{ii}} - C_{n_{oi}} C_{n_{io}}}$$

$$b_{n_o} = \frac{B_{n_o} C_{n_{ii}} - B_{n_i} C_{n_{io}}}{C_{n_{oo}} C_{n_{ii}} - C_{n_{oi}} C_{n_{io}}}$$

$$a_{n_i} = \frac{A_{n_o} C_{n_{io}} - A_{n_i} C_{n_{oo}}}{C_{n_{io}} C_{n_{io}} - C_{n_{oo}} C_{n_{ii}}}$$

$$b_{n_i} = \frac{B_{n_o} C_{n_{io}} - B_{n_i} C_{n_{oo}}}{C_{n_{oi}} C_{n_{io}} - C_{n_{oo}} C_{n_{ii}}}$$

Laminar flow

The laminar flow case may be solved by setting $E = 1$ in equation (2), which then has the general solution $R_n(\bar{r}) = C_1 \bar{r}^n + C_2 \bar{r}^{-n}$. At the walls this temperature parameter becomes

$$R_{n_i}(1) = \frac{-r^*}{2(1 - r^*) n} \frac{1}{\sinh(n \ln r^*)}$$

outer wall temperature
from inner wall heating

$$R_{n_i}(r^*) = \frac{-r^*}{2(1 - r^*) n} \coth(n \ln r^*)$$

inner wall temperature
from inner wall heating

$$R_{n_o}(1) = \frac{-1}{2(1 - r^*) n} \coth(n \ln r^*)$$

outer wall temperature
from outer wall heating

$$R_{n_o}(r^*) = \frac{-1}{2(1 - r^*) n} \frac{1}{\sinh(n \ln r^*)}$$

inner wall temperature
from outer wall heating

This is also the solution for the turbulent flow case when $Pr = 0$ since equation (2) is then independent of the velocity distribution.

Consider, for example, the case of an annular flow passage heated on the inner wall with a sinusoidal heat flux distribution around the periphery of ± 10 per cent,

$$q_{wi}'' = q_i''(1 + 0{\cdot}1 \cos \theta).$$

Employing an average heat-transfer coefficient for evaluating local wall temperature, which implies complete circumferential mixing at each cross section, we would get

$$\frac{t_{wi} - t_m}{D_H q_i''/k} = \frac{1}{Nu_i}(1 + 0{\cdot}1 \cos \theta), \qquad (5)$$

or a ± 10 per cent variation in wall temperature. On the other hand, the present analysis, equation (3b), yields

$$\frac{t_{wi} - t_m}{D_H q_i''/k} = R_{o_i}(r^*) + 0{\cdot}1\, R_{1_i}(r^*) \cos\theta$$
$$= \frac{1}{Nu_{ii}} (1 + 0{\cdot}515 \cos\theta) \text{ for } r^* = 0{\cdot}5$$
$$= \frac{1}{Nu_{ii}} (1 + 23{\cdot}4 \cos\theta) \text{ for } r^* = 0{\cdot}9 \qquad (6)$$

For fully established thermal conditions, the effect of peripheral heat flux variation on t_w is thus increased by five times for an annulus with a radius ratio of 0·5, and over 200 times for a radius ratio of 0·9.

Conduction in the tube wall will substantially reduce this effect. For air flowing in a 1-in outer tube with 0·010-in stainless steel walls (i.e. $k =$ 9 Btu/(h ft degF), wall conduction will have the following effect:

	r^*	A_{1_i}	a_{1_i}	$Nu_{ii} R_{1_i}(r^*) a_{1_i}$
Without wall conduction	0·5	0·1	0·1	0·515
With wall conduction	0·5	0·1	0·00621	0·032
Without wall conduction	0·9	0·1	0·1	23·4
With wall conduction	0·9	0·1	0·01147	2·63

The peripheral conduction in the core wall will reduce the temperature variation by a large factor. For $r^* = 0{\cdot}5$, the temperature variation is now essentially smoothed out, but for $r^* =$ 0·9, the wall temperature variation is still 26·3 times greater than what would have been predicted using an average Nusselt number.

Turbulent flow

Integration of equation (2) for turbulent flow is somewhat more involved since $E(\bar{r})$ is neither a simple nor easily determined function. Kays and Leung [2] critically reviewed the available data for application to the annulus. Since their analysis for uniform heat flux actually constitutes the fundamental eigenfunction for our problem, we will use the same relations for eddy diffusivity to calculate the higher harmonics of these eigenfunctions.

For the location of the plane of zero shear, Kays and Leung [2] found

$$\frac{\bar{s} - r^*}{1 - \bar{s}} = (r^*)^{0{\cdot}343}$$

The eddy diffusivity for momentum in the laminar sublayer is due to Diessler [6]

$$\frac{\epsilon_M}{\nu} = m u^+ y^+ [1 - \exp(-m u^+ y^+)],$$

where $m = 0{\cdot}0154$. To make the co-ordinates consistent with the annular geometry, a slightly modified co-ordinate system was used, $y_a^+ =$ $[1{\cdot}5(1 + \eta)/(1 + 2\eta^2)]\, y^+$, and the laminar sublayer was taken to extend to $y_a^+ = 42$. Reichardt's [7] middle law, with an empirical modification to fit available experimental data [2], was used in the turbulent core, the co-ordinates again made consistent with the annular geometry.

For $\bar{r} > \bar{s}$,

$$\frac{\epsilon_M}{\nu} = \frac{(1 - \bar{s})\, r_o^+}{15} (1 - \eta_o^2)(1 + 2\eta_o^2)[1 + 0{\cdot}6(\eta_o - \eta_o^2)]$$

and for $\bar{r} < \bar{s}$,

$$\frac{\epsilon_M}{\nu} = \frac{(1 - \bar{s})}{15} r_o^+ (1 - \eta_i^2)(1 + 2\eta_i^2)[1 + 0{\cdot}6\sqrt{(r^*)}(\eta_i - \eta_i^2)] \left\{1 - \left[1 - \frac{\bar{s} - r^*}{\sqrt{(r^*)}(1 - \bar{s})}\right]\eta_i\right\}$$

The ratio of eddy diffusivities, $\sigma = \epsilon_H/\epsilon_M$, is

Table 1. *Circumferential heat flux temperature functions*

$r^* = 0$

Pr	n	Re 10^4 $R_{n_o}(1)$	10^5 $R_{n_o}(1)$	10^6 $R_{n_o}(1)$
Laminar†	0	0·229		
0‡	0	0·1588	0·1462	0·1417
	1	0·500		
	2	0·250		
	3	0·1667		
	4	0·1250		
	5	0·1000		
0·01	0	0·1557	0·1123	0·0328
	1	0·490	0·346	0·0717
	2	0·245	0·1907	0·0492
	3	0·1641	0·1344	0·0410
	4	0·1220	0·1050	0·0357
	5	0·0976	0·0865	0·0330
0·03	0	0·1450	0·0629	0·01242
	1	0·444	0·1585	0·0228
	2	0·226	0·1000	0·01706
	3	0·1527	0·0776	0·01483
	4	0·1166	0·0646	0·01299
	5	0·0920	0·0568	0·01236
0·7	0	0·0316	0·00562	0·000870
	1	0·0523	0·00832	0·001181
	2	0·0395	0·00665	0·000992
	3	0·0348	0·00605	0·000916
	4	0·0321	0·00571	0·000873
	5	0·0297	0·00547	0·000843
1·0	0	0·0264	0·00451	0·000680
	1	0·0402	0·00635	0·000887
	2	0·0317	0·00520	0·000757
	3	0·0286	0·00479	0·000705
	4	0·0268	0·00455	0·000675
	5	0·0252	0·00440	0·000655
3·0	0	0·01627	0·00248	0·000345
	1	0·0201	0·00301	0·000406
	2	0·01756	0·00265	0·000362
	3	0·01664	0·00252	0·000345
	4	0·01611	0·00245	0·000335
	5	0·01561	0·00240	0·000329
10·0	0	0·01002	0·001450	0·0001917
	1	0·01110	0·001573	0·000203
	2	0·01039	0·001470	0·0001904
	3	0·01013	0·001433	0·0001853
	4	0·00998	0·001412	0·0001824
	5	0·00984	0·001398	0·0001805

† The laminar eigenfunctions are identical to $Pr = 0$ eigenfunctions for $n > 0$.
‡ The $Pr = 0$ eigenfunctions are independent of Re for $n > 0$.

Table 2. Circumferential heat flux te

$r^* = 0{\cdot}2$

		10^4				*Re*	
		$R_{n_i}(1)$	$R_{n_i}(r^*)$	$R_{n_o}(1)$	$R_{n_o}(r^*)$	$R_{n_i}(1)$	$R_{n_i}(r^*)$
Laminar†	0	0·0213	0·1177	0·205	0·1064		
0‡	0	−0·0240	0·1190	0·1712	−0·1200	−0·0248	0·1205
	1	0·0521	0·1354	0·677	0·260		
	2	0·00501	0·0627	0·314	0·0250		
	3	0·000667	0·0416	0·208	0·00333		
	4	0·0001000	0·0313	0·1563	0·0005000		
	5	0·00001600	0·0250	0·1250	0·0000800		
0·01	0	−0·0235	0·1177	0·1680	−0·1177	−0·01947	0·1031
	1	0·0513	0·1331	0·668	0·256	0·0440	0·1192
	2	0·00492	0·0616	0·309	0·0246	0·00434	0·0577
	3	0·000655	0·0410	0·205	0·00327	0·000591	0·0392
	4	0·0000983	0·0308	0·1532	0·000491	0·0000901	0·0298
	5	0·00001581	0·0247	0·1225	0·0000787	0·00001467	0·0240
0·03	0	−0·0225	0·1111	0·1608	−0·1125	−0·00985	0·0633
	1	0·0487	0·1269	0·632	0·243	0·0244	0·0751
	2	0·00468	0·0590	0·292	0·0234	0·00258	0·0416
	3	0·000624	0·0396	0·1937	0·00312	0·000375	0·0307
	4	0·0000939	0·0299	0·1452	0·000469	0·0000602	0·0245
	5	0·00001516	0·0241	0·1163	0·0000754	0·00001026	0·0205
0·7	0	−0·00214	0·0259	0·0340	0·01070	−0·000297	0·00510
	1	0·00616	0·0303	0·0846	0·0308	0·000780	0·00555
	2	0·000722	0·0222	0·0498	0·00362	0·0000900	0·00454
	3	0·0001163	0·01922	0·0407	0·000583	0·00001434	0·00420
	4	0·0000206	0·01720	0·0363	0·0001031	0·00000253	0·00400
	5	0·00000384	0·01561	0·0334	0·00001897	0·000000488	0·00385
1·0	0	−0·001437	0·0214	0·0281	−0·00725	−0·000204	0·00405
	1	0·00413	0·0244	0·0616	0·0207	0·000535	0·00436
	2	0·000492	0·01897	0·0385	0·00248	0·0000618	0·00367
	3	0·0000800	0·01690	0·0325	0·000409	0·00000985	0·00344
	4	0·00001414	0·01544	0·0296	0·0000739	0·000001745	0·00330
	5	0·00000261	0·01424	0·0280	0·00001416	0·000000344	0·00320
3·0	0	−0·000433	0·01291	0·01666	−0·00222	−0·0000615	0·00215
	1	0·001213	0·01371	0·0262	0·00611	0·0001652	0·00225
	2	0·0001485	0·01212	0·01952	0·000759	0·00001907	0·00204
	3	0·0000247	0·01146	0·01782	0·0001304	0·00000305	0·001972
	4	0·00000434	0·01094	0·01699	0·0000247	0·000000543	0·001930
	5	0·000000806	0·01047	0·01666	0·00000519	0·0000001035	0·001899
10·0	0	−0·0001328	0·00833	0·01021	−0·001000	−0·00001763	0·001250
	1	0·000341	0·00839	0·01287	0·001720	0·0000471	0·001268
	2	0·0000423	0·00794	0·01101	0·000217	0·00000544	0·001208
	3	0·00000704	0·00774	0·01054	0·0000380	0·000000871	0·001188
	4	0·000001215	0·00757	0·01031	0·00000735	0·0000001565	0·001176
	5	0·000000285	0·00740	0·01023	0·000001609	0·0000000322	0·001167

† The laminar eigenfunctions are identical to $Pr = 0$ eigenfunctions for $n > 0$
‡ The $Pr = 0$ eigenfunctions are independent of Re for $n > 0$

$r^* = 0{\cdot}2$

Re 10⁵			10⁶			
$R_{n_i}(r^*)$	$R_{n_o}(1)$	$R_{n_o}(r^*)$	$R_{n_i}(1)$	$R_{n_i}(r^*)$	$R_{n_o}(1)$	$R_{n_o}(r^*)$
0·1205	0·1639	−0·1230	−0·0247	0·1205	0·1573	−0·1230
0·1031	0·1351	−0·0974	−0·00415	0·0298	0·0406	−0·0223
0·1192	0·553	0·220	0·01088	0·0404	0·1338	0·0544
0·0577	0·259	0·0217	0·001217	0·0258	0·0701	0·00609
0·0392	0·1760	0·00296	0·0001866	0·0208	0·0534	0·000933
0·0298	0·1348	0·000451	0·0000315	0·01775	0·0452	0·0001577
0·0240	0·1097	0·0000730	0·00000562	0·01560	0·0401	0·0000279
0·0633	0·0788	−0·0488	−0·000925	0·01235	0·01667	−0·000461
0·0751	0·625	0·1217	0·00291	0·01438	0·0392	0·01451
0·0416	0·1440	0·01288	0·000336	0·01058	0·0227	0·001679
0·0307	0·1037	0·001866	0·0000535	0·00925	0·01848	0·000269
0·0245	0·0838	0·000299	0·00000939	0·00842	0·01652	0·0000480
0·0205	0·0714	0·0000505	0·000001925	0·00781	0·01533	0·00000900
0·00510	0·00606	−0·001460	−0·0000374	0·000788	0·000935	−0·0001850
0·00555	0·01259	0·00389	0·0000982	0·000859	0·001748	0·000491
0·00454	0·00810	0·000447	0·00001124	0·000731	0·001180	0·0000562
0·00420	0·00693	0·0000708	0·000001774	0·000689	0·001032	0·00000887
0·00400	0·00636	0·00001241	0·000000310	0·000663	0·000960	0·000001547
0·00385	0·00601	0·00000228	0·0000000601	0·000645	0·000915	0·000000283
0·00405	0·00485	−0·001003	−0·0000252	0·000610	0·000720	−0·0001274
0·00436	0·00928	0·00267	0·0000675	0·000662	0·001265	0·000338
0·00367	0·00620	0·000308	0·00000773	0·000574	0·000875	0·0000387
0·00344	0·00540	0·0000490	0·000001219	0·000545	0·000773	0·00000610
0·00330	0·00501	0·00000861	0·000000213	0·000527	0·000724	0·000001064
0·00320	0·00472	0·000001565	0·0000000412	0·000514	0·000693	0·0000001944
0·00215	0·00256	−0·000307	−0·00000834	0·000308	0·000362	−0·0000415
0·00225	0·00393	0·000825	0·0000219	0·000324	0·000534	0·0001095
0·00204	0·00298	0·0000950	0·00000250	0·000296	0·000407	0·00001251
0·001972	0·00273	0·00001511	0·000000394	0·000286	0·000374	0·000001970
0·001930	0·00260	0·00000266	0·0000000687	0·000281	0·000358	0·000000343
0·001899	0·00251	0·000000482	0·00000001392	0·000277	0·000347	0·0000000624
0·001250	0·001471	−0·0000900	−0·00000241	0·0001667	0·000201	−0·0000408
0·001268	0·001847	0·000235	0·00000640	0·0001727	0·000244	0·0000320
0·001208	0·001576	0·0000271	0·000000729	0·0001644	0·000207	0·00000365
0·001188	0·001505	0·00000431	0·0000001146	0·0001616	0·0001974	0·000000574
0·001176	0·001470	0·000000758	0·0000000201	0·0001600	0·0001927	0·0000000996
0·001167	0·001443	0·0000001374	0·00000000440	0·0001588	0·0001896	0·00000001810

[*Facing p.* 1192]

Table 3. Circumferential heat flux te

$r^* = 0{\cdot}5$

Pr	*n*	10⁴				1(	
		$R_{n_i}(1)$	$R_{n_i}(r^*)$	$R_{n_o}(1)$	$R_{n_o}(r^*)$	$R_{n_i}(1)$	$R_{n_i}(r^*)$
Laminar†	0	0·0428	0·1618	0·1985	0·0855		
0‡	0	−0·0496	0·1593	0·1768	−0·0987	−0·0510	0·1587
	1	0·667	0·833	1·667	1·333		
	2	0·1333	0·283	0·567	0·267		
	3	0·0423	0·1720	0·344	0·0847		
	4	0·01569	0·1260	0·252	0·0314		
	5	0·00626	0·1002	0·200	0·01251		
0·01	0	−0·0490	0·1569	0·1745	−0·0976	−0·0425	0·1370
	1	0·657	0·821	1·644	1·315	0·565	0·710
	2	0·1312	0·279	0·558	0·262	0·1139	0·246
	3	0·0416	0·1688	0·338	0·0832	0·0366	0·1518
	4	0·01538	0·1235	0·247	0·0308	0·01371	0·1129
	5	0·00612	0·0981	0·1960	0·01224	0·00553	0·0910
0·03	0	−0·0462	0·1482	0·1657	−0·0930	−0·00222	0·0832
	1	0·625	0·781	1·564	1·250	0·309	0·397
	2	0·1246	0·265	0·531	0·249	0·0634	0·1458
	3	0·0394	0·1605	0·321	0·0789	0·0209	0·0949
	4	0·01456	0·1175	0·234	0·0291	0·00806	0·0739
	5	0·00578	0·0935	0·1849	0·01158	0·00334	0·0618
0·7	0	−0·00485	0·0323	0·0353	−0·00970	−0·000678	0·00602
	1	0·0734	0·1083	0·1986	0·1496	0·00956	0·01576
	2	0·01527	0·0489	0·0785	0·0315	0·001997	0·00803
	3	0·00510	0·0369	0·0550	0·01075	0·000682	0·00649
	4	0·001990	0·0320	0·0457	0·00431	0·000355	0·00586
	5	0·000831	0·0292	0·0405	0·001850	0·000611	0·00551
1·0	0	−0·00319	0·0262	0·0287	−0·00646	−0·000460	0·00471
	1	0·0484	0·0764	0·1371	0·0995	0·00655	0·01141
	2	0·01002	0·0372	0·0574	0·0211	0·001369	0·00611
	3	0·00332	0·0294	0·0418	0·00722	0·000467	0·00505
	4	0·001279	0·0261	0·0356	0·00291	0·000238	0·00462
	5	0·000524	0·0243	0·0321	0·001254	0·000391	0·00438
3·0	0	−0·000975	0·01497	0·01652	−0·001930	−0·0001432	0·00248
	1	0·01510	0·0303	0·0471	0·0279	0·00202	0·00454
	2	0·00336	0·01837	0·0250	0·00601	0·000437	0·00290
	3	0·001241	0·01602	0·0207	0·00212	0·0001946	0·00258
	4	0·000549	0·01507	0·01909	0·000878	0·0001768	0·00244
	5	0·000264	0·01453	0·01820	0·000392	0·000256	0·00237
10·0	0	−0·00028	0·00943	0·0100	−0·000556	−0·0000411	0·001400
	1	0·00425	0·01354	0·01859	0·00774	0·000575	0·001973
	2	0·000956	0·01021	0·01250	0·001677	0·0001242	0·001508
	3	0·000360	0·00955	0·01129	0·000594	0·0000552	0·001416
	4	0·0001632	0·00929	0·01084	0·000248	0·0000499	0·001378
	5	0·0000808	0·00913	0·01060	0·0001118	0·0000722	0·001356

† The laminar eigenfunctions are identical to $Pr = 0$ eigenfunctions for $n > 0$.
‡ The $Pr = 0$ eigenfunctions are independent of Re for $n > 0$.

$r^* = 0·5$

Re 10⁵			10⁶			
$R_{n_i}(r^*)$	$R_{n_i}(1)$	$R_{n_o}(r^*)$	$R_{n_i}(1)$	$R_{n_i}(r^*)$	$R_{n_o}(1)$	$R_{n_o}(r^*)$
0·1587	0·1725	−0·1033	−0·0521	0·1587	0·1681	−0·1038
0·1370	0·1470	−0·0854	−0·00961	0·0431	0·0442	−0·01843
0·710	1·406	1·129	0·1361	0·1829	0·344	0·272
0·246	0·479	0·228	0·0283	0·0726	0·1243	0·0567
0·1518	0·293	0·0731	0·00950	0·0504	0·0811	0·01900
0·1129	0·217	0·0274	0·00374	0·0413	0·0640	0·00748
0·0910	0·1742	0·01106	0·001585	0·0360	0·0547	0·00317
0·0832	0·0862	−0·0444	−0·00254	0·01527	0·01562	−0·00509
0·397	0·768	0·617	0·0358	0·0539	0·0927	0·0720
0·1458	0·266	0·1269	0·00741	0·0248	0·0348	0·01508
0·0949	0·1671	0·0418	0·00254	0·01794	0·0235	0·00511
0·0739	0·1273	0·01612	0·001008	0·01554	0·01919	0·00204
0·0618	0·1054	0·00669	0·000436	0·01415	0·01701	0·000384
0·00602	0·00633	−0·001356	−0·0000865	0·000926	0·000962	−0·0001712
0·01576	0·0272	0·01910	0·001202	0·00224	0·00355	0·00242
0·00803	0·01180	0·00399	0·000247	0·001259	0·001599	0·000506
0·00649	0·00879	0·001348	0·0000846	0·001015	0·001218	0·0001707
0·00586	0·00762	0·000536	0·0000334	0·000935	0·001070	0·0000679
0·00551	0·00700	0·000230	0·00001441	0·000889	0·000992	0·0000292
0·00471	0·00500	−0·000981	−0·0000582	0·000705	0·000746	−0·0001198
0·01141	0·01931	0·01310	0·001191	0·001624	0·00252	0·001663
0·00611	0·00874	0·00274	0·000464	0·000944	0·001179	0·000348
0·00505	0·00668	0·000924	0·0000586	0·000776	0·000917	0·0001173
0·00462	0·00587	0·000367	0·0000232	0·000720	0·000815	0·0000467
0·00438	0·00544	0·0001577	0·00000996	0·000688	0·000762	0·0000200
0·00248	0·00260	−0·000286	−0·00001905	0·000348	0·000366	−0·0000397
0·00454	0·00703	0·00405	0·000270	0·000633	0·000930	0·000540
0·00290	0·00376	0·000846	0·0000564	0·000414	0·000494	0·0001124
0·00258	0·00312	0·000285	0·00001900	0·000370	0·000407	0·0000376
0·00244	0·00287	0·0001127	0·00000751	0·000352	0·000371	0·00001473
0·00237	0·00273	0·0000479	0·00000323	0·000342	0·000351	0·00000619
0·001400	0·001471	−0·0000825	−0·00000556	0·0001852	0·0001987	−0·00001130
0·001973	0·00272	0·001151	0·0000794	0·000270	0·000360	0·0001576
0·001508	0·001795	0·000241	0·00001648	0·000206	0·000232	0·0000328
0·001416	0·001613	0·0000811	0·00000555	0·0001929	0·000207	0·00001097
0·001378	0·001540	0·0000320	0·00000219	0·0001876	0·0001963	0·00000429
0·001356	0·001499	0·00001362	0·000000941	0·0001846	0·0001905	0·000001803

Table 4. Circumferential heat flu

$r^* = 0·8$

Pr	n	10^4					
		$R_{n_i}(1)$	$R_{n_i}(r^*)$	$R_{n_o}(1)$	$R_{n_o}(r^*)$	$R_{n_i}(1)$	$R_{n_i}(r^*)$
Laminar†	0	0·0570	0·1790	0·1910	0·0717		
0‡	0	0·0671	0·1702	0·1770	−0·0834	−0·0692	0·1690
	1	8·889	9·111	11·389	11·111		
	2	2·168	2·398	2·984	2·710		
	3	0·925	1·140	1·425	1·157		
	4	0·492	0·702	0·877	0·615		
	5	0·294	0·496	0·620	0·367		
0·01	0	−0·0662	0·1680	0·1740	0·0815	−0·0581	0·1470
	1	8·775	8·994	11·243	10·968	7·561	7·751
	2	2·140	2·356	2·946	2·675	1·846	2·035
	3	0·913	1·125	1·406	1·141	0·789	0·974
	4	0·485	0·692	0·865	0·607	0·421	0·601
	5	0·289	0·489	0·611	0·362	0·252	0·427
0·03	0	−0·0636	0·1613	0·1640	−0·0771	−0·0305	0·0876
	1	8·260	8·472	10·625	10·357	4·153	4·265
	2	2·013	2·222	2·788	2·524	1·016	1·127
	3	0·858	1·063	1·334	1·075	0·436	0·545
	4	0·455	0·655	0·822	0·571	0·233	0·341
	5	0·271	0·464	0·583	0·340	0·1404	0·245
0·7	0	−0·00685	0·0351	0·0357	−0·00856	−0·000962	0·00636
	1	0·985	1·026	1·297	1·252	0·1284	0·1354
	2	0·241	0·282	0·352	0·307	0·0315	0·0384
	3	0·1037	0·1440	0·1769	0·1323	0·01353	0·0205
	4	0·0556	0·0956	0·1154	0·0713	0·00727	0·01415
	5	0·0335	0·0729	0·0868	0·0432	0·00440	0·01120
1	0	−0·00456	0·0282	0·0287	−0·00564	−0·000655	0·00495
	1	0·648	0·681	0·865	0·830	0·0880	0·0934
	2	0·1587	0·1908	0·239	0·204	0·0216	0·0270
	3	0·0681	0·1000	0·1225	0·0879	0·00927	0·01465
	4	0·0365	0·0681	0·0818	0·0474	0·00499	0·01032
	5	0·0219	0·0533	0·0628	0·0288	0·00301	0·00830
3·0	0	−0·001353	0·01588	0·01632	−0·001715	−0·000204	0·00259
	1	0·1996	0·216	0·250	0·232	0·0271	0·0298
	2	0·0493	0·0656	0·0752	0·0569	0·00664	0·00934
	3	0·0215	0·0378	0·0429	0·0246	0·00286	0·00555
	4	0·01180	0·0280	0·0315	0·01333	0·001540	0·00422
	5	0·00732	0·0235	0·0262	0·00813	0·000939	0·00360
10·0	0	−0·000390	0·00980	0·01000	−0·000500	−0·0000582	0·001443
	1	0·0559	0·0656	0·0749	0·0643	0·00770	0·00917
	2	0·01384	0·0236	0·0264	0·01581	0·001887	0·00335
	3	0·00606	0·01577	0·01743	0·00684	0·000812	0·00227
	4	0·00334	0·01303	0·01428	0·00371	0·000438	0·001895
	5	0·00208	0·01176	0·01281	0·00227	0·000267	0·001719

† The laminar eigenfunctions are identical to $Pr = 0$ eigenfunctions for $n > 0$.
‡ The $Pr = 0$ eigenfunctions are independent of Re for $n > 0$.

ferential heat flux temperature functions

$r^* = 0.8$

Re						
10^5			10^6			
$R_{n_i}(r^*)$	$R_{n_o}(1)$	$R_{n_o}(r^*)$	$R_{n_i}(1)$	$R_{n_i}(r^*)$	$R_{n_i}(1)$	$R_{n_o}(r^*)$
0·1690	0·1740	−0·0870	−0·0699	0·1673	0·1710	−0·0884
0·1470	0·1487	−0·0725	−0·01362	0·0461	0·0476	−0·01760
7·751	10·285	10·036	1·835	1·891	2·358	2·293
2·035	2·696	2·450	0·449	0·506	0·627	0·562
0·974	1·289	1·047	0·1932	0·249	0·306	0·241
0·601	0·794	0·558	0·1037	0·1588	0·1929	0·1297
0·427	0·562	0·334	0·0626	0·1166	0·1403	0·0783
0·0876	0·0901	−0·0390	−0·00348	0·01640	0·01612	−0·00452
4·265	5·342	5·207	0·485	0·506	0·625	0·606
1·127	1·407	1·274	0·1188	0·1398	0·1675	0·1487
0·545	0·678	0·547	0·0510	0·0718	0·0827	0·0640
0·341	0·421	0·293	0·0274	0·0480	0·0529	0·0344
0·245	0·302	0·1762	0·01650	0·0368	0·0390	0·0208
0·00636	0·00641	−0·001186	−0·0001197	0·000954	0·000981	−0·0001522
0·1354	0·1679	0·1603	0·01622	0·01740	0·0214	0·0203
0·0384	0·0468	0·0393	0·00397	0·00513	0·00602	0·00497
0·0205	0·0244	0·01690	0·001703	0·00286	0·00318	0·00214
0·01415	0·01652	0·00909	0·000912	0·00206	0·00218	0·001150
0·01120	0·01284	0·00549	0·000548	0·001688	0·001718	0·000695
0·00495	0·00508	−0·000822	−0·0000835	0·000741	0·000752	−0·0001038
0·0934	0·1158	0·1100	0·01115	0·01203	0·01475	0·01395
0·0270	0·0328	0·0270	0·00273	0·00361	0·00422	0·00342
0·01465	0·01737	0·01160	0·001171	0·00204	0·00227	0·001471
0·01032	0·01197	0·00623	0·000627	0·001494	0·001580	0·000791
0·00830	0·00944	0·00377	0·000377	0·001237	0·001260	0·000478
0·00259	0·00262	−0·000251	−0·0000271	0·000364	0·000370	−0·0000338
0·0298	0·0367	0·0339	0·00361	0·00405	0·00491	0·00453
0·00934	0·01108	0·00833	0·000877	0·001319	0·001487	0·001113
0·00555	0·00636	0·00359	0·000380	0·000814	0·000852	0·000479
0·00422	0·00467	0·001938	0·000216	0·000638	0·000630	0·000259
0·00360	0·00390	0·001177	0·0001544	0·000558	0·000527	0·0001568
0·001443	0·001492	−0·0000735	−0·00000788	0·0001941	0·0001970	−0·00000990
0·00917	0·01114	0·0965	0·001053	0·001269	0·001520	0·001323
0·00335	0·00386	0·00237	0·000256	0·000471	0·000521	0·000325
0·00227	0·00251	0·001022	0·0001109	0·000324	0·000336	0·0001399
0·001895	0·00204	0·000551	0·0000631	0·000272	0·000271	0·0000754
0·001719	0·001820	0·000335	0·0000448	0·000249	0·000241	0·0000457

Table 5. Circumferential heat flux tem

$r^* = 0{\cdot}9$

Pr	n	10^4				10^5	
		$R_{n_i}(1)$	$R_{n_i}(r^*)$	$R_{n_o}(1)$	$R_{n_o}(r^*)$	$R_{n_i}(1)$	$R_{n_i}(r^*)$
Laminar†	0	0·0609	0·1830	0·1880	0·0677		
0‡	0	−0·0713	0·1729	0·1763	−0·0793	−0·0739	0·1707
	1	42·63	42·87	47·63	47·37		
	2	10·60	10·84	12·04	11·78		
	3	4·67	4·90	5·45	5·19		
	4	2·59	2·83	3·14	2·88		
	5	1·63	1·86	2·08	1·81		
0·01	0	−0·0701	0·1703	0·1732	−0·0778	−0·0618	0·1482
	1	42·090	42·325	47·026	46·766	36·311	36·514
	2	10·464	10·697	11·871	11·611	9·029	9·232
	3	4·608	4·840	5·373	5·115	3·978	4·179
	4	2·558	2·789	3·097	2·840	2·210	2·410
	5	1·611	1·839	2·044	1·7896	1·392	1·591
0·03	0	−0·0670	0·1627	0·1640	−0·0735	−0·0330	0·0894
	1	40·121	40·344	44·830	44·582	19·861	19·980
	2	9·975	10·197	11·343	11·095	4·941	5·060
	3	4·392	4·614	5·132	4·885	2·178	2·296
	4	2·439	2·659	2·957	2·712	1·211	1·329
	5	1·535	1·754	1·9485	1·7057	0·764	0·881
0·7	0	−0·00738	0·0356	0·0358	−0·00825	−0·001030	0·00641
	1	4·7471	4·7896	5·3889	5·3448	0·6181	0·6253
	2	1·1812	1·2237	1·377	1·333	0·1538	0·1610
	3	0·5209	0·5633	0·633	0·589	0·0679	0·0750
	4	0·2898	0·3321	0·372	0·328	0·0378	0·0449
	5	0·1829	0·2251	0·2504	0·2067	0·0239	0·0310
1·0	0	−0·00489	0·0284	0·0286	−0·00542	−0·000706	0·00501
	1	3·1224	3·1559	3·5749	3·5406	0·4237	0·4293
	2	0·7768	0·8103	0·9185	0·8840	0·1054	0·1110
	3	0·3424	0·3759	0·425	0·391	0·0465	0·0520
	4	0·1904	0·2238	0·252	0·218	0·0259	0·0314
	5	0·1201	0·1534	0·1711	0·1370	0·01636	0·0219
3·0	0	−0·001460	0·01617	0·01640	−0·001632	−0·000217	0·00261
	1	0·9671	0·9838	1·0254	1·0076	0·1305	0·1333
	2	0·2412	0·2579	0·2619	0·2442	0·0325	0·0352
	3	0·1068	0·1234	0·1254	0·1079	0·01433	0·01708
	4	0·0597	0·0764	0·0776	0·0602	0·00798	0·01072
	5	0·0380	0·0546	0·0568	0·0391	0·00504	0·00778
10·0	0	−0·000418	0·00985	0·00995	−0·000473	−0·0000948	0·001455
	1	0·2711	0·2810	0·2904	0·2800	0·03710	0·03858
	2	0·0677	0·0775	0·0780	0·0677	0·00923	0·01071
	3	0·0300	0·0398	0·0401	0·0299	0·00407	0·00555
	4	0·01679	0·0267	0·0269	0·01666	0·00227	0·00375
	5	0·01067	0·0205	0·0212	0·01086	0·001432	0·00291

* The laminar eigenfunctions are identical to $Pr = 0$ eigenfunctions for $n > 0$.

† The $Pr = 0$ eigenfunctions are independent of Re for $n > 0$.

$r^* = 0{\cdot}9$

Re 10⁵			10⁶			
$R_{n_i}(r^*)$	$R_{n_o}(1)$	$R_{n_o}(r^*)$	$R_{n_i}(1)$	$R_{n_i}(r^*)$	$R_{n_o}(1)$	$R_{n_o}(r^*)$
0·1707	0·1732	−0·0830	−0·0753	0·1700	0·1718	−0·0846
0·1482	0·1491	−0·0693	−0·01460	0·0463	0·0470	−0·01660
36·514	40·543	40·319	8·840	8·899	9·885	9·822
9·232	10·255	10·031	2·200	2·259	2·508	2·444
4·179	4·924	4·687	0·970	1·030	1·142	1·078
2·410	2·840	2·604	0·540	0·599	0·664	0·600
1·591	1·765	1·546	0·341	0·400	0·442	0·379
0·0894	0·0905	−0·0372	−0·00382	0·01635	0·01623	−0·00433
19·980	22·330	22·201	2·341	2·361	2·620	2·600
5·060	5·674	5·544	0·583	0·603	0·6674	0·6472
2·296	2·574	2·444	0·257	0·277	0·3073	0·2873
1·329	1·488	1·360	0·1431	0·1632	0·1799	0·1599
0·881	0·981	0·854	0·0904	0·1104	0·1209	0·1004
0·00641	0·00644	−0·001141	−0·0001287	0·00961	0·000975	−0·0001451
0·6253	0·6946	0·6871	0·0782	0·0793	0·0880	0·0869
0·1610	0·1782	0·1708	0·01947	0·02055	0·02268	0·02164
0·0750	0·0829	0·0754	0·00859	0·00967	0·01058	0·00955
0·0449	0·0494	0·0420	0·00478	0·00587	0·00635	0·00532
0·0310	0·0340	0·0265	0·00302	0·00410	0·00445	0·00335
0·00501	0·00508	−0·000786	−0·0000900	0·000744	0·000750	−0·0000998
0·4293	0·4771	0·4713	0·0538	0·0546	0·0606	0·0597
0·1110	0·1229	0·1172	0·01338	0·01421	0·01570	0·01487
0·0520	0·0575	0·0518	0·00590	0·00673	0·00739	0·00656
0·0314	0·0346	0·0288	0·00329	0·00411	0·00448	0·00365
0·0219	0·0239	0·0182	0·00207	0·00290	0·00313	0·00231
0·00261	0·00263	−0·000243	−0·0000291	0·000367	0·000370	−0·0000324
0·1333	0·1479	0·1451	0·01745	0·01785	0·01975	0·01937
0·0352	0·0389	0·0361	0·00434	0·00474	0·00520	0·00482
0·01708	0·01874	0·01600	0·001915	0·00231	0·00250	0·00213
0·01072	0·01165	0·00892	0·001066	0·001459	0·001560	0·001184
0·00778	0·00841	0·00560	0·000673	0·001065	0·001123	0·000748
0·001455	0·001482	−0·0000706	−0·00000846	0·0001952	0·0001970	−0·00000946
0·03858	0·04281	0·04129	0·00509	0·00531	0·00584	0·00564
0·01071	0·01178	0·01028	0·001266	0·001481	0·001595	0·001410
0·00555	0·00604	0·00455	0·000558	0·000772	0·000799	0·000642
0·00375	0·00402	0·00254	0·000310	0·000524	0·000514	0·000358
0·00291	0·003112	0·001594	0·0001959	0·000409	0·000411	0·000218

due to Jenkins [8], but Jenkins' values were increased by a factor 1·20 to bring them more in line with experimental data. This ratio, σ, is taken as 1 in the sublayers. Equation (2) was integrated numerically on a digital computer. The values calculated for the eigenfunctions for the first five harmonics at the inner and outer walls are tabulated in Tables 1 through 5.

The example considered above for laminar flow will have the following results at a Reynolds number of 10^5 and a Prandtl number of 0·7.

$$q''_{wi} = q''_i \, (1 + 0{\cdot}1 \cos \theta).$$

Perfect circumferential mixing

$$\frac{t_{wi}(\theta) - t_m}{D_H \, q''_i / k} = \frac{1}{Nu_{ii}} (1 + 0{\cdot}1 \cos \theta).$$

Present solution, equation (3b)

$$\frac{t_{wi} - t_m}{D_H \, q''_i / k} = R_{o_i}(r^*) + 0{\cdot}1 \, R_{1_i}(r^*) \cos \theta$$

$$= \frac{1}{Nu_{ii}} (1 + 0{\cdot}262 \cos \theta) \text{ for } r^* = 0{\cdot}5$$

$$= \frac{1}{Nu_{ii}} (1 + 9{\cdot}75 \cos \theta) \text{ for } r^* = 0{\cdot}9$$

The wall conduction of the 0·010-in wall mentioned above will reduce these effects to

$$= \frac{1}{Nu_{ii}} (1 + 0{\cdot}1583 \cos \theta) \text{ for } r^* = 0{\cdot}5$$

and

$$= \frac{1}{Nu_{ii}} (1 + 2{\cdot}95 \cos \theta) \text{ for } r^* = 0{\cdot}9.$$

Turbulent mixing and wall conduction greatly reduce the temperature variation, but the annulus with $r^* = 0{\cdot}9$ still produces a ± 295 per cent wall temperature variation for a ± 10 per cent wall heat flux variation.

DISCUSSION AND CONCLUSIONS

Wall heat flux variation around the periphery of the flow channel tends to stratify the flow, resulting in hot layers along the high flux side and cooler layers along the low flux side. Estimates of wall temperature variation based on the average Nusselt number and the mixed mean fluid temperature then imply a "perfect" circumferential mixing of the flow. The present analysis takes into consideration the resistance to circumferential mixing, based on the as yet unproven assumption that in turbulent flow the eddy conductivity in the circumferential direction is the same as in the radial direction.

The effect of variable wall heat flux has been shown [4] to be quite significant for the circular tube, even in turbulent flow. For the annulus we have shown that the problem is more severe, being aggravated by the geometry itself. Mixing must primarily take place around the circumference since the radial path through the center is blocked; this geometry effect is expressed by the parameter r_o/D_H which ranges from 0·5 for the circular tube to 5·0 for a $r^* = 0{\cdot}9$ annulus.

The long circumferential conduction path, especially as r^* approaches 1·00, suggests that the thermal entry length necessary to reach the fully developed thermal conditions postulated in this analysis may be very much greater than the thermal entry length required for symmetrical heating. An estimate of this entry length problem may be made by considering the flow passage to be divided by radial barriers into a number of parallel channels. If there is no conduction between these channels, the fluid temperature in each will rise in proportion to the heat rate applied at its solid boundary, and the temperature at its solid boundary can be estimated from an average heat-transfer coefficient and this fluid temperature. This is, in essence, the case of zero circumferential mixing, and the flow length required to reach a wall temperature variation of the magnitude predicted by equation (3) is the minimum thermal entry length.

In the example considered above, of turbulent flow with conduction in the wall, this minimum entry length is about 180 hydraulic diameters for $r^* = 0{\cdot}5$, and 6760 hydraulic diameters for $r^* = 0{\cdot}9$. Thus equation (3) represents an upper bound on the magnitude of wall temperature variation, and how closely this bound is approached in a practical case will depend on the radius ratio and Reynolds number.

Although the illustrative examples presented consider only the case of the inner wall heated with one harmonic of peripheral heat flux variation, it should be again emphasized that the theory is sufficiently general to include both walls heated (or one heated and one cooled) with any peripheral heat flux distribution that can be adequately expressed by a Fourier expansion with five harmonics.

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Résumé—On présente une analyse du transport de chaleur dans un tube annulaire à noyau coaxial avec un flux de chaleur imposé arbitrairement autour de la périphérie de l'une des parois ou de toutes les deux. Des solutions ont été obtenues pour des conditions entièrement établies hydrodynamiquement et thermiquement pour un flux de chaleur constant par unité de longueur du tube, à la fois pour les régimes d'écoulement laminaire et turbulent. Avec ces résultats, la variation de température autour de chaque paroi qui s'ensuit peut être prédite. Contrairement à ce que l'on pourrait attendre, la variation de température pariétale est très importante en écoulement turbulent aussi bien qu'en écoulement laminaire. Un exemple montre l'importance de cet effet.

Zusammenfassung—Für den Ringraum von konzentrisch angeordneten Kreisrohren wird eine Analyse des Wärmeübergangs dargelegt bei beliebig vorgegebener Wärmestromdichte für die eine Rohrwand oder für beide Rohrwände. Es ergaben sich Lösungen für den hydrodynamisch und thermisch voll ausgebildeten Zustand sowohl der laminaren als auch der turbulenten Lösung bei konstantem Wärmestrom pro Rohrlängeneinheit. Mit diesen Ergebnissen kann die sich ergebende Temperaturänderung rings um eine Rohrwand vorhergesagt werden. Im Gegensatz zu den Erwartungen ist die Änderung der Wandtemperatur ganz beträchtlich sowohl bei turbulenter als auch bei laminarer Strömung. Ein Beispiel zeigt die Bedeutung dieses Effektes.